The measurement and integration of stream velocity (and almost anything else) using a remarkable formula

Alternative Hydraulics Paper 10

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A traditional two-point formula for integrating velocity measurements in streams is shown to be surprisingly accurate, and better than other hydrometric formulae that require more measurements. It is just as accurate for any function or data that are smooth enough to be able to be represented by a polynomial of up to sixth degree[1].

1 Introduction

Throughout hydrometry it is common practice, when measuring the discharge in a stream, just to measure the velocity at two points, at 20% and 80% of the total depth and say that the mean of those two is the integrated mean over the whole depth. Thus one writes $\tilde{u}$, the mean velocity, as:

$$\tilde{u} = \tfrac{1}{2}\left(u_{0.2} + u_{0.8}\right). \tag{1}$$

I have always been contemptuous of this, as I knew something of the intricacies of numerical approximation, especially as in a turbulent shear flow (also known as a river ...) the velocity gradient at the bed becomes very large and a simple polynomial, the basis of integration formulae, is not good enough.

2 The approximation of velocity measurements

A simple but systematic approach is to assume that the velocity u is a combination of a function that shows infinite velocity gradient at the bed plus other simple polynomial terms, as used elsewhere throughout science and engineering:

$$u(Z) = \sum_{n=\nu,0}^{N} c_n Z^n = c_\nu Z^\nu + c_0 + c_1 Z + c_2 Z^2 + \ldots + c_N Z^N, \tag{2}$$

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where $Z = z/h$ is dimensionless elevation above the bed, h is the depth, and where $\nu \approx 1/7$ for smooth boundaries and $\approx 1/5$ for rough ones. The Z^ν for $\nu < 1$ shows the required infinite gradient as $Z \to 0$; it can be shown to be an excellent approximation to the better-known logarithmic expression. The c_n, for $n = \nu, 0, \ldots, N$ are coefficients to be found from a particular data set by optimisation. For this channel flow application, we require $u = 0$ at $Z = 0$, so that $c_0 = 0$.

Consider a number of laboratory results from the 1980s and 90s when laser measurement methods became feasible and fashionable, shown in Figure 1. The solid lines show that the approximation (2) works very well for $N = 3$, and even for the black dashed line with just $N = 2$ it works to within practical accuracy.

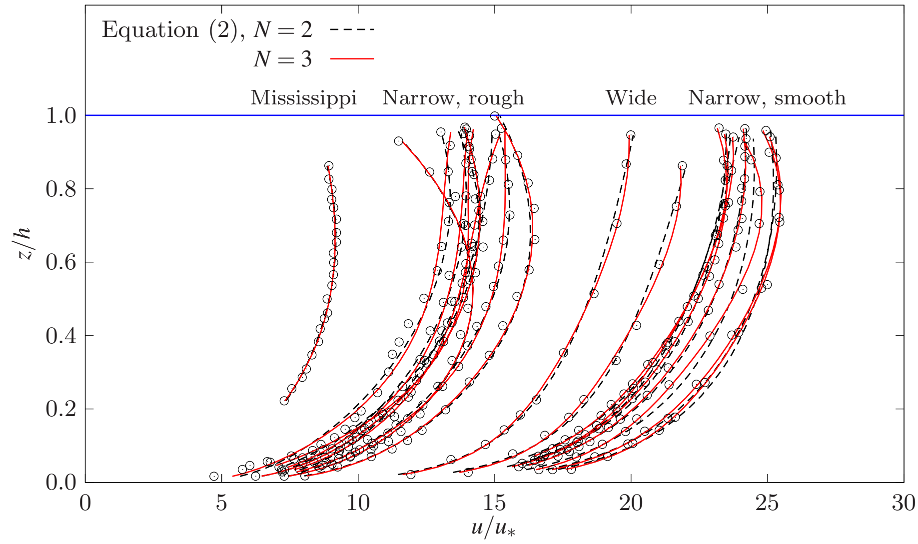

Figure 1: Experimental and field profile measurements

3 Integrating to find the mean

In hydrometry (and probably other areas) the spatial mean in the vertical is required to calculate the overall discharge Here, having determined the coefficients c_n, the mean is simply obtained by integrating equation (2):

$$\tilde{u} = \int_0^1 \left(\sum_{n=\nu,0}^{N} c_n Z^n \right) \mathrm{d}Z = \sum_{n=\nu,0}^{N} \frac{c_n}{n+1}. \tag{3}$$

I applied that to each of the cases shown in Figure 1, obtaining a numerical value in each case. Then, the point of the exercise, I went to test the simple equation (1) to show how it, and hence hydrometric practice of the last century, was wrong, assuming that I could use equation (2) to obtain $u_{0.2}$ and $u_{0.8}$. *I found to my great surprise that equation (1) was accurate to within 0.5% in all of the eighteen cases shown* – including with a strong velocity maximum present, *and* without that. Why?

4 Surprising accuracy of the simple formula

The answer is remarkably simple, general, and very surprising. Consider the approximating equation (2), made up of monomial terms Z^n. Now consider the integral of that general term from 0 to 1: $1\,(n+1)$ as shown in equation (3). Now consider applying the scheme equation (1) to approximating that integral of Z^n: $\frac{1}{2}(0.2^n + 0.8^n)$. It looks nothing like $1/(n+1)$! However, we form the error of the approximation:

$$e = \tfrac{1}{2}(0.2^n + 0.8^n) - \frac{1}{n+1}, \tag{4}$$

and plot it as a function of n from $\nu = 1/7$ as used for flow over smooth boundaries through $\nu = 1/5$ for rough boundaries, and then for integer n, as in the series above, to $n = N = 6$.

The results are shown in Figure 2 and are remarkable. For $n \leq 2$, the errors are $< 0.7\%$. For $n \leq 6$, the errors are less than 1% or just greater for $n = 6$. For the real data of Figure 1 the author found an accuracy of $< 0.5\%$. That simple formula is remarkably accurate.

What it also means is, that when combined as in equation (2), evaluating *any* function or data that is sufficiently smooth so as to be able to be approximated by a low-degree polynomial, can be simply integrated by taking the mean at the 20% and 80% levels.

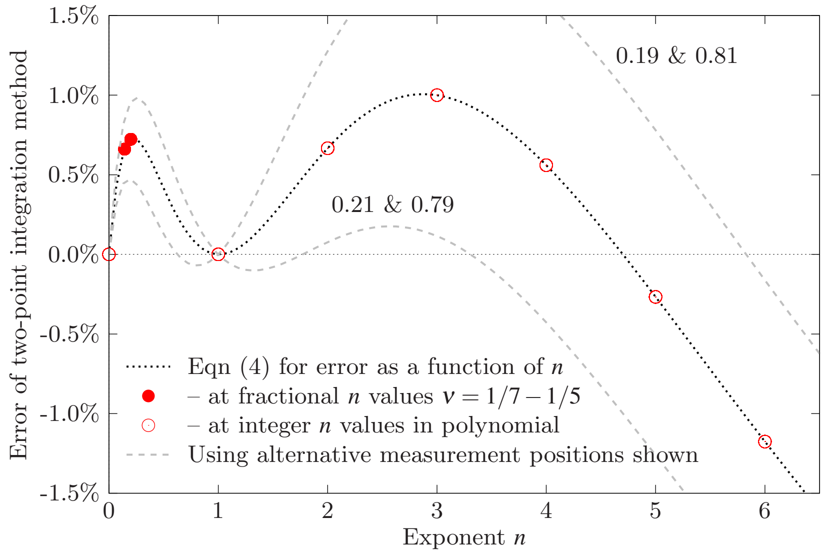

Figure 2: The error of approximating the integral of Z^n by the Two-point 20/80 method

Figure 2 also contains some additional results two grey dashed lines corresponding to a test, varying the numerical values of 0.2 and 0.8 by 0.01, giving $\frac{1}{2}(0.19^n + 0.81^n)$ and $\frac{1}{2}(0.21^n + 0.79^n)$. They are not as good as the original and the best!

5 Conclusions

The hydrographers' two-point 20/80 method, equation (1), is a very good approximation to the integral of velocity over depth, with an error of less than 1%, which is rather less than what has been believed (3.5% in the WMO Handbook) *and better than all other methods that require more measurements*. Its combination of efficiency, just two points, and accuracy of 1%, make it the method to be recommended.

Without the turbulent shear term $n = \nu$ the approximating function (2) is simply a conventional polynomial of degree N, such as used throughout science and engineering, and so, with the addition of a c_0 term that was unnecessary above, equation (1) could be used rather more widely than in stream gauging.

Here, a somewhat facetious comment can be made. In 1906, the Italian economist Vilfredo Pareto noted that approximately 80% of Italy's land was owned by 20% of the population. This has led others to name the eponymous Principle (also termed the 80–20 rule).which states that roughly 80% of consequences come from 20% of causes. Of course this is quite a different matter from our considerations, but the numerical coincidence was striking, even the precision of the 80% and 20% that was found here.